

## Solvable kinetic Gaussian model in an external field

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In this paper, the single-spin transition dynamics is used to investigate the kinetic Gaussian model in a periodic external field. We first derive the fundamental dynamic equations, and then treat an isotropic  $d$ -dimensional hypercubic lattice Gaussian spin system with Fourier's transformation method. We obtain exactly the local magnetization and the equal-time pair-correlation function. The critical characteristics of the dynamical relaxation  $\tau_{\mathbf{q}}$ , the complex susceptibility  $\chi(\omega, \mathbf{q})$ , and the dynamical response are discussed. The results show that the time evolution of the dynamical quantities and the dynamical responses of the system strongly depend on the frequency and the wave vector of the external field.

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### I. INTRODUCTION

The purpose of this work is to investigate the dynamical behavior of a cooperative spin system, particularly, the dynamical critical behavior. It is well known that a great progress in the understanding of critical dynamics has been made, since Glauber [1] and Kawasaki [2] completed their pioneering work on the time-dependent Ising model. In the past two decades much research has been devoted to a better understanding of dynamical behavior of various systems, and many theoretical methods have been applied and developed [3–19]. Among these studies, the main attention has been focused on the discrete spin systems, such as the Ising model and the Potts model, and a little on the continuous symmetry  $O(n)$  spin systems. Nevertheless, as far as our knowledge goes, only a few analytical results were presented. In the present work and the previous paper [20], we are focusing on obtaining the exact analytical results. This is our main motivation.

The Gaussian model is a variation of the Ising model. It is a uniaxial continuous spin model that shows different static critical behavior from the Ising model. Although its static critical properties have been investigated clearly, little attention has been paid to dynamical critical behavior. This is also the reason we study the kinetic Gaussian model. Within the framework of Glauber dynamics in our previous paper [20], we have obtained dynamical critical exponent  $z = 1/\nu = 2$  at the critical point  $K_c = b/2d$  based on rigorous analytical derivation.

To our knowledge, only the kinetic Ising model with time-dependent external field has been investigated in detail [21,22]. The present work is attempting to investigate further the dynamic behavior of the kinetic Gaussian model with time-dependent external field. This paper is organized as follows: In Sec. II, we first summarize the basic theory of the single-spin transition critical dynamics, and then derive the fundamental equations of the kinetic Gaussian model in a

periodic external field. In Sec. III, an isotropic  $d$ -dimensional hypercubic lattice Gaussian spin model is treated by Fourier's transformation. We exactly obtain the local magnetization and the equal-time spin-pair-correlation function. The critical characteristics of the dynamical relaxation  $\tau_{\mathbf{q}}$ , the complex susceptibility  $\chi(\omega, \mathbf{q})$ , and the dynamical response of the system to the time-dependent external field are investigated. Finally, we end up the paper with concluding remarks in Sec. IV.

### II. FUNDAMENTAL EQUATIONS

#### A. Basic theory of the single-spin transition critical dynamics

A single-spin transition critical dynamics based on Glauber's theory [1], applying to both discrete-spin and continuous-spin systems, was presented in our previous paper [20]. For the sake of application here we only give a summary.

Spins system with Hamiltonian,  $\mathcal{H}(\{\sigma_i\})$ , where  $\sigma_i$  is the spin of site  $i$  and can take discrete values or continuous values, interacts with a large heat bath with temperature  $T$ . The heat bath gives rise to spontaneous transition of spins via exchange of the energy. The probability of transition of the  $i$ th spin per unit time from one value  $\sigma_i$  to another possible value  $\hat{\sigma}_i$  is denoted by  $W_i(\sigma_i \rightarrow \hat{\sigma}_i)$ . Under the assumption of single-spin transition, the probability distribution function  $P(\{\sigma_{ij}\}, t)$  of the system, being in the configuration  $(\sigma_1, \sigma_2, \dots, \sigma_N)$  at time  $t$ , is governed by the master equation

$$\frac{d}{dt} P(\{\sigma_{ij}\}, t) = \sum_i \sum_{\hat{\sigma}_i} [-W_i(\sigma_i \rightarrow \hat{\sigma}_i) P(\{\sigma_{ij}\}, t) + W_i(\hat{\sigma}_i \rightarrow \sigma_i) P(\{\sigma_{j \neq i}\}, \hat{\sigma}_i, t)], \quad (1)$$

where the spin transition probability satisfies the following restrictive conditions:

(1) Ergodicity:

$$\forall \sigma_j, \hat{\sigma}_j: W_j(\sigma_j \rightarrow \hat{\sigma}_j) \neq 0; \quad (2)$$

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(2) Positivity:

$$\forall \sigma_j, \hat{\sigma}_j: W_j(\sigma_j \rightarrow \hat{\sigma}_j) \geq 0; \quad (3)$$

(3) Normalization:

$$\forall \sigma_j: \sum_{\hat{\sigma}_j} W_j(\sigma_j \rightarrow \hat{\sigma}_j) = 1; \quad (4)$$

(4) Detailed balance:

$$\forall \sigma_j, \hat{\sigma}_j: \frac{W_j(\sigma_j \rightarrow \hat{\sigma}_j)}{W_j(\hat{\sigma}_j \rightarrow \sigma_j)} = \frac{P_{\text{eq}}(\sigma_1, \dots, \hat{\sigma}_j, \dots, \sigma_N)}{P_{\text{eq}}(\sigma_1, \dots, \sigma_j, \dots, \sigma_N)}, \quad (5)$$

in which

$$P_{\text{eq}}(\{\sigma\}) = \frac{1}{Z} \exp[-\beta \mathcal{H}(\{\sigma\})], \quad Z = \sum_{\{\sigma\}} \exp[-\beta \mathcal{H}(\{\sigma\})], \quad (6)$$

where  $P_{\text{eq}}$  is the equilibrium Boltzmann distribution function,  $Z$  the partition function and  $\mathcal{H}(\{\sigma\})$  the Hamiltonian of the system.

By use of both the master Eq. (1) and the normalized condition (4), the time-evolving equations of the local magnetization and the equal-time spin-pair-correlation function can be expressed as, respectively

$$\begin{aligned} \frac{d}{dt} \langle \sigma_k(t) \rangle &= -\langle \sigma_k(t) \rangle + \sum_{\{\sigma_j\}} \left( \sum_{\hat{\sigma}_k} \hat{\sigma}_k W_k(\sigma_k \rightarrow \hat{\sigma}_k) \right) \\ &\quad \times P(\{\sigma_j\}, t), \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{d}{dt} \langle \sigma_k(t) \sigma_l(t) \rangle &= -2 \langle \sigma_k(t) \sigma_l(t) \rangle + \sum_{\{\sigma_j\}} \left[ \sigma_k(t) \right. \\ &\quad \times \left( \sum_{\hat{\sigma}_l} \hat{\sigma}_l W_l(\sigma_l \rightarrow \hat{\sigma}_l) \right) + \sigma_l(t) \\ &\quad \times \left. \left( \sum_{\hat{\sigma}_k} \hat{\sigma}_k W_k(\sigma_k \rightarrow \hat{\sigma}_k) \right) \right] P(\{\sigma_j\}, t), \end{aligned} \quad (8)$$

where

$$\langle \sigma_k(t) \rangle = \sum_{\{\sigma_j\}} \sigma_k(t) P(\{\sigma_j\}, t), \quad (9)$$

$$\langle \sigma_k(t) \sigma_l(t) \rangle = \sum_{\{\sigma_j\}} \sigma_k(t) \sigma_l(t) P(\{\sigma_j\}, t). \quad (10)$$

Although the evolution starts with a certain initial state at  $t=0$ , the system must be relaxed toward the final objective, which is an equilibrium state characterized by  $P_{\text{eq}} = (1/Z) \exp[-\beta \mathcal{H}(\{\sigma\})]$  in the absence of time-dependent external field, via interaction with heat bath. In addition, it is usually considered that the transition probabilities of the individual spins depend merely on the momentary values of the neighboring spins as well as the influence of the heat bath. So, even if the transition probability cannot be derived exactly by means of microscope, the following form:

$$W_i(\sigma_i \rightarrow \hat{\sigma}_i) = \frac{1}{Q_i} \exp \left[ -\beta \mathcal{H}_i \left( \hat{\sigma}_i, \sum_{\langle ij \rangle} \sigma_j \right) \right], \quad (11)$$

is well chosen, where  $Q_i$  is the coefficient determined by the normalized condition (4). Equation (11) means that the transition probability from  $\sigma_j$  to  $\hat{\sigma}_j$  only depends on the heat Boltzmann factor of the neighboring spins. If the system is in a periodic low-frequency external field, Eq. (11) is still a possible choice.

## B. The model and the fundamental equations

To study further the dynamical behavior of the Gaussian spins system near the critical point, we put the system in a periodic low-frequency external field which may be regarded as an electromagnetic wave [22]. The reduced Hamiltonian of the system under consideration is

$$-\beta \mathcal{H} = \sum_{\langle ij \rangle} K_{ij} \sigma_i \sigma_j + \sum_i h_i(t) \sigma_i, \quad (12)$$

where

$$\beta = \frac{1}{k_\beta T}, \quad K_{ij} = \frac{J_{ij}}{k_\beta T}, \quad h_i(t) = \frac{H_i(t)}{k_\beta T} = \frac{H_0}{k_\beta T} \exp(i\omega t - iqx_i), \quad (13)$$

the first sum goes over all nearest-neighbor pairs of lattice and the second over all sites. Unlike Ising spin model, the Gaussian model have two extensions [23]: first, the spin can take any real value in the range of  $(-\infty, +\infty)$ ; second, to prevent all spins from tending to infinity, the probability of finding a given spin between  $\sigma_i$  and  $\sigma_i + d\sigma_i$  is assumed to be the Gaussian-type distribution

$$f(\sigma_i) d\sigma_i = \sqrt{\frac{b}{2\pi}} \exp \left[ -\frac{b}{2} \sigma_i^2 \right] d\sigma_i, \quad (14)$$

where  $b$  is a distribution constant independent of temperature.

In terms of those mentioned above, we can derive the fundamental equations of the kinetic Gaussian model in the external field. Following Eq. (11), we choose the spin-transition probability as

$$\begin{aligned} W_i(\sigma_i \rightarrow \hat{\sigma}_i) &= \frac{1}{Q_i} \exp \left[ \sum_w K_{i,i+w} \hat{\sigma}_i \sigma_{i+w} + h_i \hat{\sigma}_i \right] \\ &= \frac{1}{Q_i} \exp[E_i \hat{\sigma}_i], \end{aligned} \quad (15)$$

where

$$E_i = \sum_w K_{i,i+w} \sigma_{i+w} + h_i, \quad (16)$$

and  $\sum_w$  means the summation over nearest neighbors. As the spin variable takes continuous values, the summation for spin turns into the integration

$$\sum_\sigma \rightarrow \int_{-\infty}^{\infty} f(\sigma) d\sigma, \quad (17)$$

then the normalized factor  $Q_i$  can be determined as

$$Q_i = \sum_{\hat{\sigma}_i} \exp(E_i \hat{\sigma}_i) = \int \exp(E_i \hat{\sigma}_i) f(\hat{\sigma}_i) d\hat{\sigma}_i = \exp\left(\frac{E_i^2}{2b}\right), \quad (18)$$

and another useful combination formula can also be written as

$$\sum_{\hat{\sigma}_i} \hat{\sigma}_i W_i(\sigma_i \rightarrow \hat{\sigma}_i) = \int \hat{\sigma}_i W_i(\sigma_i \rightarrow \hat{\sigma}_i) f(\hat{\sigma}_i) d\hat{\sigma}_i = E_i/b. \quad (19)$$

Substituting Eq. (19) into the time-evolving Eqs. (7),(8), one gets

$$\frac{d}{dt} \langle \sigma_i(t) \rangle = -\langle \sigma_i(t) \rangle + \frac{1}{b} \sum_w K_{i,i+w} \langle \sigma_{i+w}(t) \rangle + \frac{1}{b} h_i(t), \quad (20)$$

$$\begin{aligned} \frac{d}{dt} \langle \sigma_i(t) \sigma_j(t) \rangle &= -2 \langle \sigma_i(t) \sigma_j(t) \rangle \\ &+ \frac{1}{b} \sum_w [K_{j,j+w} \langle \sigma_i(t) \sigma_{j+w}(t) \rangle \\ &+ K_{i,i+w} \langle \sigma_{i+w}(t) \sigma_j(t) \rangle] + \frac{1}{b} [h_i(t) \\ &\times \langle \sigma_j(t) \rangle + h_j(t) \langle \sigma_i(t) \rangle]. \end{aligned} \quad (21)$$

Equations (20) and (21) are the fundamental equations of the kinetic Gaussian model in the external field.

### III. EXACT SOLUTION

For a  $d$ -dimensional isotropic hypercubic lattice, the dynamic equations of the local magnetization (20) and the spin-pair-correlation function (21) can be rewritten as

$$\begin{aligned} \frac{d}{dt} \langle \sigma_i(t) \rangle &= -\langle \sigma_i(t) \rangle + \frac{1}{b} \sum_{\mathbf{r}_1} K(\mathbf{r}_i - \mathbf{r}_1) \langle \sigma_1(t) \rangle \\ &+ \frac{\beta H_0}{b} e^{i\omega t - i\mathbf{q} \cdot \mathbf{r}_i}, \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{d}{dt} \langle \sigma_i(t) \sigma_j(t) \rangle &= -2 \langle \sigma_i(t) \sigma_j(t) \rangle + \frac{1}{b} \left[ \sum_{\mathbf{r}_1} K(\mathbf{r}_i - \mathbf{r}_1) \right. \\ &\times \langle \sigma_1(t) \sigma_j(t) \rangle + \sum_{\mathbf{r}_1} K(\mathbf{r}_1 - \mathbf{r}_j) \\ &\times \langle \sigma_i(t) \sigma_1(t) \rangle \left. \right] + \frac{\beta H_0}{b} [\langle \sigma_i(t) \rangle e^{i\omega t - i\mathbf{q} \cdot \mathbf{r}_j} \\ &+ \langle \sigma_j(t) \rangle e^{i\omega t - i\mathbf{q} \cdot \mathbf{r}_i}], \end{aligned} \quad (23)$$

where

$$\begin{aligned} \mathbf{q} &= (q_1, q_2, \dots, q_d), \quad \mathbf{r} = (x_1, x_2, \dots, x_d), \\ \mathbf{i} &= (i_1, i_2, \dots, i_d), \quad \mathbf{j} = (j_1, j_2, \dots, j_d), \dots, \end{aligned} \quad (24)$$

and

$$K(\mathbf{r}_i - \mathbf{r}_j) = \begin{cases} K, & \text{nearest-neighbor-pair,} \\ 0, & \text{others.} \end{cases} \quad (25)$$

Introducing Fourier's transformation

$$\langle M(\mathbf{q}', t) \rangle = \sum_{\mathbf{i}} \langle \sigma_i(t) \rangle e^{i\mathbf{q}' \cdot \mathbf{r}_i}, \quad (26)$$

$$\langle G(\mathbf{q}', \mathbf{q}'', t) \rangle = \sum_{\mathbf{k}, \mathbf{l}} \langle \sigma_{\mathbf{k}}(t) \sigma_{\mathbf{l}}(t) \rangle e^{i\mathbf{q}' \cdot \mathbf{r}_k + i\mathbf{q}'' \cdot \mathbf{r}_l}, \quad (27)$$

which satisfy

$$\frac{1}{N} \sum_{\mathbf{r}_1} e^{i(\mathbf{q} - \mathbf{q}') \cdot \mathbf{r}_1} = \delta_{\mathbf{q}, \mathbf{q}'}, \quad (28)$$

$$\frac{1}{N} \sum_{\mathbf{q}} e^{i\mathbf{q} \cdot (\mathbf{r}_1 - \mathbf{r}_k)} = \delta_{\mathbf{l}, \mathbf{k}}, \quad (29)$$

we have

$$\frac{d}{dt} \langle M(\mathbf{q}', t) \rangle + \left[ 1 - \frac{1}{b} K(\mathbf{q}') \right] \langle M(\mathbf{q}', t) \rangle = \frac{NH_0\beta}{b} \delta_{\mathbf{q}, \mathbf{q}'} e^{i\omega t}, \quad (30)$$

$$\begin{aligned} \frac{d}{dt} \langle G(\mathbf{q}', \mathbf{q}'', t) \rangle &+ \left[ \left( 1 - \frac{1}{b} K(\mathbf{q}') \right) + \left( 1 - \frac{1}{b} K(\mathbf{q}'') \right) \right] \\ &\times \langle G(\mathbf{q}', \mathbf{q}'', t) \rangle = \frac{NH_0\beta}{b} [\langle M(\mathbf{q}', t) \rangle \delta_{\mathbf{q}, \mathbf{q}''} \\ &+ \langle M(\mathbf{q}'', t) \rangle \delta_{\mathbf{q}, \mathbf{q}'}] e^{i\omega t}, \end{aligned} \quad (31)$$

where

$$\begin{aligned} K(\mathbf{q}) &= \sum_{\mathbf{i}} K(\mathbf{r}_i - \mathbf{r}_j) e^{i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \\ &= K \sum_{i=1}^d (e^{iq_i a} + e^{-iq_i a}) = 2K \sum_{i=1}^d \cos(q_i a), \end{aligned} \quad (32)$$

$$K(0) = 2Kd, \quad (33)$$

and  $a$  is the lattice constant,  $q_i$  is the  $i$ th component of the wave vector  $\mathbf{q}$ , and  $d$  is the spatial dimensionality.

First we solve Eq. (30). Obviously, it is a first-order linear inhomogeneous differential equation with the canonical form

$$\frac{dy(t)}{dt} + P(t)y(t) = Q(t),$$

and its general solution is

$$y(t) = \frac{1}{\mu(t)} \left[ \mu(t_0) y(t_0) + \int_{t_0}^t \mu(\xi) Q(\xi) d\xi \right],$$

where

$$\mu(t) = \exp \left[ \int P(t) dt \right].$$

$$\tau_{\mathbf{q}'} = \frac{1}{1 - (1/b)K(\mathbf{q}')} = \frac{1}{1 - (2K/b) \sum_{i=1}^d \cos(q'_i a)} \quad (35)$$

Applying it to Eq. (30), one can get the following exact solution:

and

$$\begin{aligned} \langle M(\mathbf{q}', t) \rangle &= \exp \left( -\frac{t}{\tau_{\mathbf{q}'}} \right) \left[ \langle M(\mathbf{q}', 0) \rangle \right. \\ &\quad \left. + \frac{NH_0\beta}{b} \delta_{\mathbf{q}, \mathbf{q}'} \int_0^t \exp \left( \frac{\xi}{\tau_{\mathbf{q}'}} + i\omega\xi \right) d\xi \right] \\ &= [\langle M(\mathbf{q}', 0) \rangle - H_0\chi(\omega, \mathbf{q}') \delta_{\mathbf{q}, \mathbf{q}'}] \exp \left( -\frac{t}{\tau_{\mathbf{q}'}} \right) \\ &\quad + H_0\chi(\omega, \mathbf{q}') \delta_{\mathbf{q}, \mathbf{q}'} e^{i\omega t}, \end{aligned} \quad (34)$$

$$\chi(\omega, \mathbf{q}') = \frac{N\beta}{b} \cdot \frac{1}{1 - (2K/b) \sum_{i=1}^d \cos(q'_i a) + i\omega} \quad (36)$$

are the wave-number-dependent relaxation time and the frequency- and wave-number-dependent complex susceptibility, respectively.

From expression (35) we can see that  $\tau_{\mathbf{q}'}$  is finite for  $\mathbf{q}' \neq 0$  as the temperature approaches the critical point  $T_c$  ( $K_c = J/k_\beta T_c = b/2d$ ), while it becomes to infinity for  $\mathbf{q}' = 0$ .

Substituting the solution (34) of the local magnetization into Eq. (31), the time evolution equation of the spin-pair-correlation can be rewritten as

where

$$\begin{aligned} \frac{d}{dt} \langle G(\mathbf{q}', \mathbf{q}'', t) \rangle &+ \left[ \left( 1 - \frac{1}{b} K(\mathbf{q}') \right) + \left( 1 - \frac{1}{b} K(\mathbf{q}'') \right) \right] \langle G(\mathbf{q}', \mathbf{q}'', t) \rangle \\ &= \frac{NH_0\beta}{b} [\langle M(\mathbf{q}', 0) \rangle - H_0\chi(\omega, \mathbf{q}') \delta_{\mathbf{q}, \mathbf{q}'}] \delta_{\mathbf{q}, \mathbf{q}''} \exp \left( -\frac{t}{\tau_{\mathbf{q}'}} + i\omega t \right) + \frac{NH_0\beta}{b} [\langle M(\mathbf{q}'', 0) \rangle \\ &\quad - H_0\chi(\omega, \mathbf{q}'') \delta_{\mathbf{q}, \mathbf{q}''}] \delta_{\mathbf{q}, \mathbf{q}'} \exp \left( -\frac{t}{\tau_{\mathbf{q}''}} + i\omega t \right) + \frac{NH_0\beta}{b} [H_0\chi(\omega, \mathbf{q}') + H_0\chi(\omega, \mathbf{q}'')] \delta_{\mathbf{q}, \mathbf{q}'} \delta_{\mathbf{q}, \mathbf{q}''} e^{2i\omega t}. \end{aligned} \quad (37)$$

Equation (37) is also a first-order linear inhomogeneous differential equation. One can give its general solution

$$\begin{aligned} \langle G(\mathbf{q}', \mathbf{q}'', t) \rangle &= \exp \left( -\frac{t}{\tau_{\mathbf{q}'}} - \frac{t}{\tau_{\mathbf{q}''}} \right) \left\{ \langle G(\mathbf{q}', \mathbf{q}'', 0) \rangle + \frac{NH_0\beta}{b} [\langle M(\mathbf{q}', 0) \rangle - H_0\chi(\omega, \mathbf{q}') \delta_{\mathbf{q}, \mathbf{q}'}] \delta_{\mathbf{q}, \mathbf{q}''} \int_0^t \exp \left( \frac{\xi}{\tau_{\mathbf{q}'}} + i\omega\xi \right) d\xi \right. \\ &\quad \left. + \frac{NH_0\beta}{b} [\langle M(\mathbf{q}'', 0) \rangle - H_0\chi(\omega, \mathbf{q}'') \delta_{\mathbf{q}, \mathbf{q}''}] \delta_{\mathbf{q}, \mathbf{q}'} \int_0^t \exp \left( \frac{\xi}{\tau_{\mathbf{q}'}} + i\omega\xi \right) d\xi + \frac{NH_0\beta}{b} [H_0\chi(\omega, \mathbf{q}') \right. \\ &\quad \left. + H_0\chi(\omega, \mathbf{q}'')] \delta_{\mathbf{q}, \mathbf{q}'} \delta_{\mathbf{q}, \mathbf{q}''} \times \int_0^t \exp \left( \frac{\xi}{\tau_{\mathbf{q}'}} + \frac{\xi}{\tau_{\mathbf{q}''}} + 2i\omega\xi \right) d\xi \right\} \\ &= [\langle G(\mathbf{q}', \mathbf{q}'', 0) \rangle - \langle M(\mathbf{q}', 0) \rangle H_0\chi(\omega, \mathbf{q}'') \delta_{\mathbf{q}, \mathbf{q}''} - \langle M(\mathbf{q}'', 0) \rangle H_0\chi(\omega, \mathbf{q}') \delta_{\mathbf{q}, \mathbf{q}'} \\ &\quad + H_0^2\chi(\omega, \mathbf{q}') \chi(\omega, \mathbf{q}'') \delta_{\mathbf{q}, \mathbf{q}'} \delta_{\mathbf{q}, \mathbf{q}''}] \exp \left( -\frac{t}{\tau_{\mathbf{q}'}} - \frac{t}{\tau_{\mathbf{q}''}} \right) + [\langle M(\mathbf{q}', 0) \rangle - H_0\chi(\omega, \mathbf{q}') \delta_{\mathbf{q}, \mathbf{q}'}] H_0\chi(\omega, \mathbf{q}'') \delta_{\mathbf{q}, \mathbf{q}''} \\ &\quad \times \exp \left( -\frac{t}{\tau_{\mathbf{q}'}} + i\omega t \right) + [\langle M(\mathbf{q}'', 0) \rangle - H_0\chi(\omega, \mathbf{q}'') \delta_{\mathbf{q}, \mathbf{q}''}] H_0\chi(\omega, \mathbf{q}') \delta_{\mathbf{q}, \mathbf{q}'} \exp \left( -\frac{t}{\tau_{\mathbf{q}''}} + i\omega t \right) \\ &\quad + H_0^2\chi(\omega, \mathbf{q}') \chi(\omega, \mathbf{q}'') \delta_{\mathbf{q}, \mathbf{q}'} \delta_{\mathbf{q}, \mathbf{q}''} e^{2i\omega t}. \end{aligned} \quad (38)$$

To make the solution an explicit one, we note that the factor  $\exp(-t/\tau_{\mathbf{q}})$  can be rewritten as

$$\exp \left( -\frac{t}{\tau_{\mathbf{q}}} \right) = \exp \left[ - \left( 1 - \frac{2K}{b} \sum_{i=1}^d \cos(q_i a) \right) t \right] = e^{-t} \prod_{i=1}^d \exp \left( \frac{2K}{b} t \frac{e^{iq_i a} + e^{-iq_i a}}{2} \right)$$

in which

$$\exp\left(\frac{2K}{b}t \frac{e^{iq_i a} + e^{-iq_i a}}{2}\right)$$

is just the generating function for the first-kind of imaginary argument Bessel function

$$\exp\left[\frac{x}{2}(\lambda + \lambda^{-1})\right] = \sum_{\alpha=-\infty}^{\infty} \lambda^{\alpha} I_{\alpha}(x), \quad (39)$$

where  $I_{\alpha}(x)$  is the first kind of imaginary argument Bessel function. Hence

$$\exp\left(-\frac{t}{\tau_{\mathbf{q}}}\right) = e^{-t} \prod_{i=1}^d \sum_{n_i=-\infty}^{\infty} (e^{iq_i a})^{n_i} I_{n_i}\left(\frac{2K}{b}t\right) = e^{-t} \prod_{i=1}^d \sum_{n_i=-\infty}^{\infty} e^{iq_i x_{n_i}} I_{n_i}\left(\frac{2K}{b}t\right) = e^{-t} \sum_{\mathbf{n}} e^{i\mathbf{q} \cdot \mathbf{r}_{\mathbf{n}}} I_{n_1}\left(\frac{2K}{b}t\right) \cdots I_{n_d}\left(\frac{2K}{b}t\right) \quad (40)$$

for convenience, where the summations for  $n_1, \dots, n_d$  from  $-\infty$  to  $\infty$  are denoted by  $\Sigma_{\mathbf{n}}$ . Then Eqs. (34) and (38) can be rewritten as

$$\langle M(\mathbf{q}', t) \rangle = [\langle M(\mathbf{q}', 0) \rangle - H_0 \chi(\omega, \mathbf{q}') \delta_{\mathbf{q}, \mathbf{q}'}] \times e^{-t} \sum_{\mathbf{n}} e^{i\mathbf{q}' \cdot \mathbf{r}_{\mathbf{n}}} I_{n_1}\left(\frac{2K}{b}t\right) \cdots I_{n_d}\left(\frac{2K}{b}t\right) + H_0 \chi(\omega, \mathbf{q}') \delta_{\mathbf{q}, \mathbf{q}'} e^{i\omega t}, \quad (41)$$

$$\begin{aligned} \langle G(\mathbf{q}', \mathbf{q}'', t) \rangle &= e^{-2t} [\langle G(\mathbf{q}', \mathbf{q}'', 0) \rangle - \langle M(\mathbf{q}', 0) \rangle H_0 \chi(\omega, \mathbf{q}'') \delta_{\mathbf{q}, \mathbf{q}''} - \langle M(\mathbf{q}'', 0) \rangle H_0 \chi(\omega, \mathbf{q}') \delta_{\mathbf{q}, \mathbf{q}'} \\ &\quad + H_0^2 \chi(\omega, \mathbf{q}') \chi(\omega, \mathbf{q}'') \delta_{\mathbf{q}, \mathbf{q}'} \delta_{\mathbf{q}, \mathbf{q}''}] \times \sum_{\mathbf{n}, \mathbf{m}} e^{i\mathbf{q}' \cdot \mathbf{r}_{\mathbf{n}} + i\mathbf{q}'' \cdot \mathbf{r}_{\mathbf{m}}} I_{n_1}\left(\frac{2K}{b}t\right) I_{m_1}\left(\frac{2K}{b}t\right) \cdots I_{n_d}\left(\frac{2K}{b}t\right) I_{m_d}\left(\frac{2K}{b}t\right) \\ &\quad + e^{-t} e^{i\omega t} [\langle M(\mathbf{q}', 0) \rangle - H_0 \chi(\omega, \mathbf{q}') \delta_{\mathbf{q}, \mathbf{q}'}] H_0 \chi(\omega, \mathbf{q}'') \delta_{\mathbf{q}, \mathbf{q}''} \times \sum_{\mathbf{n}} e^{i\mathbf{q}' \cdot \mathbf{r}_{\mathbf{n}}} I_{n_1}\left(\frac{2K}{b}t\right) \cdots I_{n_d}\left(\frac{2K}{b}t\right) \\ &\quad + e^{-t} e^{i\omega t} [\langle M(\mathbf{q}'', 0) \rangle - H_0 \chi(\omega, \mathbf{q}'') \delta_{\mathbf{q}, \mathbf{q}''}] H_0 \chi(\omega, \mathbf{q}') \delta_{\mathbf{q}, \mathbf{q}'} \times \sum_{\mathbf{n}} e^{i\mathbf{q}'' \cdot \mathbf{r}_{\mathbf{n}}} I_{n_1}\left(\frac{2K}{b}t\right) \cdots I_{n_d}\left(\frac{2K}{b}t\right) \\ &\quad + H_0^2 \chi(\omega, \mathbf{q}') \chi(\omega, \mathbf{q}'') \delta_{\mathbf{q}, \mathbf{q}'} \delta_{\mathbf{q}, \mathbf{q}''} e^{2i\omega t}. \end{aligned} \quad (42)$$

Taking the inverse Fourier transformation

$$\langle \sigma_{\mathbf{k}}(t) \rangle = \frac{1}{N} \sum_{\mathbf{q}'} \langle M(\mathbf{q}', t) \rangle e^{-i\mathbf{q}' \cdot \mathbf{r}_{\mathbf{k}}}, \quad (43)$$

$$\langle \sigma_{\mathbf{k}}(t) \sigma_{\mathbf{l}}(t) \rangle = \frac{1}{N^2} \sum_{\mathbf{q}', \mathbf{q}''} \langle G(\mathbf{q}', \mathbf{q}'', t) \rangle e^{-i\mathbf{q}' \cdot \mathbf{r}_{\mathbf{k}} - i\mathbf{q}'' \cdot \mathbf{r}_{\mathbf{l}}}, \quad (44)$$

and using the following relation:

$$\frac{1}{N} \sum_{\mathbf{q}} e^{i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)} = \delta_{ij}, \quad (45)$$

the local magnetization and the pair correlation of the  $d$ -dimensional hypercubic system can be written as

$$\begin{aligned} \langle \sigma_{\mathbf{k}}(t) \rangle &= e^{-t} \sum_{\mathbf{n}} \left\{ \langle \sigma_{\mathbf{n}}(0) \rangle I_{k_1 - n_1}\left(\frac{2K}{b}t\right) \cdots I_{k_d - n_d}\left(\frac{2K}{b}t\right) - \frac{1}{N} H_0 \chi(\omega, \mathbf{q}) e^{i\mathbf{q} \cdot \mathbf{r}_{\mathbf{n}}} I_{k_1 + n_1}\left(\frac{2K}{b}t\right) \cdots I_{k_d + n_d}\left(\frac{2K}{b}t\right) \right\} \\ &\quad + \frac{1}{N} H_0 \chi(\omega, \mathbf{q}) e^{i\omega t - i\mathbf{q} \cdot \mathbf{r}_{\mathbf{k}}}, \end{aligned} \quad (46)$$

and

$$\begin{aligned}
\langle \sigma_{\mathbf{k}}(t) \sigma_{\mathbf{l}}(t) \rangle &= e^{-2t} \sum_{\mathbf{n}, \mathbf{m}} \langle \sigma_{\mathbf{n}}(0) \sigma_{\mathbf{m}}(0) \rangle I_{k_1-n_1} \left( \frac{2K}{b} t \right) I_{l_1-m_1} \left( \frac{2K}{b} t \right) \cdots I_{k_d-n_d} \left( \frac{2K}{b} t \right) I_{l_d-m_d} \left( \frac{2K}{b} t \right) \\
&\quad - \frac{1}{N} e^{-2t} H_0 \chi(\omega, \mathbf{q}) e^{-i\mathbf{q} \cdot \mathbf{r}_1} \sum_{\mathbf{n}, \mathbf{m}} \langle \sigma_{\mathbf{n}}(0) \rangle e^{i\mathbf{q} \cdot \mathbf{r}_m} \times I_{k_1-n_1} \left( \frac{2K}{b} t \right) I_{m_1} \left( \frac{2K}{b} t \right) \cdots I_{k_d-n_d} \left( \frac{2K}{b} t \right) I_{m_d} \left( \frac{2K}{b} t \right) \\
&\quad - \frac{1}{N} e^{-2t} H_0 \chi(\omega, \mathbf{q}) e^{-i\mathbf{q} \cdot \mathbf{r}_k} \sum_{\mathbf{n}, \mathbf{m}} \langle \sigma_{\mathbf{m}}(0) \rangle e^{i\mathbf{q} \cdot \mathbf{r}_n} \times I_{n_1} \left( \frac{2K}{b} t \right) I_{l_1-m_1} \left( \frac{2K}{b} t \right) \cdots I_{n_d} \left( \frac{2K}{b} t \right) I_{l_d-m_d} \left( \frac{2K}{b} t \right) \\
&\quad + \frac{1}{N^2} e^{-2t} H_0^2 \chi^2(\omega, \mathbf{q}) \sum_{\mathbf{n}, \mathbf{m}} e^{i\mathbf{q} \cdot \mathbf{r}_{n+m}} \times I_{k_1+n_1} \left( \frac{2K}{b} t \right) I_{l_1+m_1} \left( \frac{2K}{b} t \right) \cdots I_{k_d+n_d} \left( \frac{2K}{b} t \right) I_{l_d+m_d} \left( \frac{2K}{b} t \right) \\
&\quad + \frac{2}{N} e^{-t} e^{i\omega t} H_0 \chi(\omega, \mathbf{q}) e^{-i\mathbf{q} \cdot \mathbf{r}_1} \sum_{\mathbf{n}} \langle \sigma_{\mathbf{n}}(0) \rangle I_{k_1-n_1} \left( \frac{2K}{b} t \right) \cdots I_{k_d-n_d} \left( \frac{2K}{b} t \right) \\
&\quad - \frac{2}{N^2} e^{-t} e^{i\omega t} H_0^2 \chi^2(\omega, \mathbf{q}) e^{-i\mathbf{q} \cdot \mathbf{r}_1} \sum_{\mathbf{n}} e^{i\mathbf{q} \cdot \mathbf{r}_n} I_{k_1+n_1} \left( \frac{2K}{b} t \right) \cdots I_{k_d+n_d} \left( \frac{2K}{b} t \right) + \frac{1}{N^2} H_0^2 \chi^2(\omega, \mathbf{q}) e^{2i\omega t - i\mathbf{q} \cdot \mathbf{r}_{k+1}},
\end{aligned} \tag{47}$$

respectively. Because  $n_i$  (or  $m_i$ ) can take any real value in the region  $(-\infty, \infty)$ , the summations for  $n_i$  and  $-n_i$  (or for  $m_i$  and  $-m_i$ ) are equal. In addition, the summation indexes  $\mathbf{n}$  and  $\mathbf{m}$  can exchange each other. Then, the Eqs. (46) and (47) can be rewritten as

$$\begin{aligned}
\langle \sigma_{\mathbf{k}}(t) \rangle &= e^{-t} \sum_{\mathbf{n}} \left[ \langle \sigma_{\mathbf{n}}(0) \rangle - \frac{1}{N} H_0 \chi(\omega, \mathbf{q}) e^{i\mathbf{q} \cdot \mathbf{r}_n} \right] I_{k_1-n_1} \left( \frac{2K}{b} t \right) \cdots I_{k_d-n_d} \left( \frac{2K}{b} t \right) + \frac{1}{N} H_0 \chi(\omega, \mathbf{q}) e^{i\omega t - i\mathbf{q} \cdot \mathbf{r}_k}, \tag{48} \\
\langle \sigma_{\mathbf{k}}(t) \sigma_{\mathbf{l}}(t) \rangle &= e^{-2t} \sum_{\mathbf{n}, \mathbf{m}} \left[ \langle \sigma_{\mathbf{n}}(0) \sigma_{\mathbf{m}}(0) \rangle + \frac{1}{N^2} H_0^2 \chi^2(\omega, \mathbf{q}) e^{-i\mathbf{q} \cdot \mathbf{r}_{n-m}} \right] \\
&\quad \times I_{k_1-n_1} \left( \frac{2K}{b} t \right) I_{l_1-m_1} \left( \frac{2K}{b} t \right) \cdots I_{k_d-n_d} \left( \frac{2K}{b} t \right) I_{l_d-m_d} \left( \frac{2K}{b} t \right) - \frac{1}{N} e^{-2t} H_0 \chi(\omega, \mathbf{q}) [e^{-i\mathbf{q} \cdot \mathbf{r}_k} + e^{-i\mathbf{q} \cdot \mathbf{r}_l}] \\
&\quad \times \sum_{\mathbf{n}, \mathbf{m}} \langle \sigma_{\mathbf{n}}(0) \rangle e^{i\mathbf{q} \cdot \mathbf{r}_m} I_{k_1-n_1} \left( \frac{2K}{b} t \right) I_{m_1} \left( \frac{2K}{b} t \right) \cdots I_{k_d-n_d} \left( \frac{2K}{b} t \right) I_{m_d} \left( \frac{2K}{b} t \right) \\
&\quad + \frac{2}{N} e^{-t} e^{i\omega t} H_0 \chi(\omega, \mathbf{q}) e^{-i\mathbf{q} \cdot \mathbf{r}_1} \sum_{\mathbf{n}} [\langle \sigma_{\mathbf{n}}(0) \rangle - H_0 \chi(\omega, \mathbf{q}) e^{-i\mathbf{q} \cdot \mathbf{r}_n}] \times I_{k_1-n_1} \left( \frac{2K}{b} t \right) \cdots I_{k_d-n_d} \left( \frac{2K}{b} t \right) \\
&\quad + \frac{1}{N^2} H_0^2 \chi^2(\omega, \mathbf{q}) e^{2i\omega t - i\mathbf{q} \cdot \mathbf{r}_{k+1}}, \tag{49}
\end{aligned}$$

where  $\langle \sigma_{\mathbf{n}}(0) \rangle$  and  $\langle \sigma_{\mathbf{n}}(0) \sigma_{\mathbf{m}}(0) \rangle$  correspond to their initial values.

So far we have obtained the exact solutions of the  $d$ -dimensional kinetic Gaussian model in the periodic external field. Letting  $H_0=0$ , one can get

$$\begin{aligned}
\langle \sigma_{k_1 \dots k_d}(t) \rangle &= e^{-t} \sum_{n_1, \dots, n_d = -\infty}^{\infty} \langle \sigma_{n_1 \dots n_d}(0) \rangle I_{k_1-n_1} \left( \frac{2K}{b} t \right) \cdots I_{k_d-n_d} \left( \frac{2K}{b} t \right), \tag{50} \\
\langle \sigma_{k_1 \dots k_d}(t) \sigma_{l_1 \dots l_d}(t) \rangle &= e^{-2t} \sum_{n_1, \dots, n_d = -\infty}^{\infty} \sum_{m_1, \dots, m_d = -\infty}^{\infty} \langle \sigma_{n_1 \dots n_d}(0) \sigma_{m_1 \dots m_d}(0) \rangle \\
&\quad \times I_{k_1-n_1} \left( \frac{2K}{b} t \right) I_{l_1-m_1} \left( \frac{2K}{b} t \right) \cdots I_{k_d-n_d} \left( \frac{2K}{b} t \right) I_{l_d-m_d} \left( \frac{2K}{b} t \right). \tag{51}
\end{aligned}$$

Equations (50) and (51) are just the exact solutions of the  $d$ -dimensional kinetic Gaussian model in zero external field, which agree with the results obtained in our previous paper [20].

To understand the physical meaning of the Eqs. (50) and (51), it is interesting to investigate the long-time asymptotic behavior of the local magnetization and the spin-pair-correlation function. By use of the asymptotic expansion expression of the first kind of imaginary argument Bessel function

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} \frac{(-)^n (v, n)}{(2x)^n} + \frac{e^{-x+(v+\frac{1}{2})\pi i}}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} \frac{(v, n)}{(2x)^n}, \quad (-\pi/2 < \arg x < 3\pi/2), |x| \rightarrow \infty, \quad (52)$$

where

$$(v, n) = \frac{\Gamma\left(\frac{1}{2} + v + n\right)}{n! \Gamma\left(\frac{1}{2} + v - n\right)},$$

one can get

$$\langle \sigma_{\mathbf{k}}(t) \rangle \sim \sum_{\mathbf{n}} \left[ \langle \sigma_{\mathbf{n}}(0) \rangle - \frac{1}{N} H_0 \chi(\omega, \mathbf{q}) e^{i\mathbf{q} \cdot \mathbf{r}_{\mathbf{n}}} \right] \frac{1}{t^{d/2}} e^{-t/\tau} + \frac{1}{N} H_0 \chi(\omega, \mathbf{q}) e^{i\omega t - i\mathbf{q} \cdot \mathbf{r}_{\mathbf{k}}}, \quad (53)$$

$$\begin{aligned} \langle \sigma_{\mathbf{k}}(t) \sigma_{\mathbf{l}}(t) \rangle \sim & \sum_{\mathbf{n}, \mathbf{m}} \left[ \langle \sigma_{\mathbf{n}}(0) \sigma_{\mathbf{m}}(0) \rangle + \frac{1}{N^2} H_0^2 \chi^2(\omega, \mathbf{q}) e^{-i\mathbf{q} \cdot \mathbf{r}_{\mathbf{n}-\mathbf{m}}} - \frac{1}{N} H_0 \chi(\omega, \mathbf{q}) (e^{-i\mathbf{q} \cdot \mathbf{r}_{\mathbf{k}-\mathbf{m}} + e^{-i\mathbf{q} \cdot \mathbf{r}_{\mathbf{l}-\mathbf{m}}}) \langle \sigma_{\mathbf{n}}(0) \rangle \right] \frac{1}{t^d} e^{-2t/\tau} \\ & + \frac{2}{N} H_0 \chi(\omega, \mathbf{q}) e^{i\omega t - i\mathbf{q} \cdot \mathbf{r}_{\mathbf{l}}} \sum_{\mathbf{n}} \left[ \langle \sigma_{\mathbf{n}}(0) \rangle - H_0 \chi(\omega, \mathbf{q}) e^{-i\mathbf{q} \cdot \mathbf{r}_{\mathbf{n}}} \right] \frac{1}{t^{d/2}} e^{-t/\tau} + \frac{1}{N^2} H_0^2 \chi^2(\omega, \mathbf{q}) e^{2i\omega t - i\mathbf{q} \cdot \mathbf{r}_{\mathbf{k}+\mathbf{l}}}, \quad (54) \end{aligned}$$

where

$$\tau = \frac{1}{1 - 2kd/b}, \quad (55)$$

is the relaxation time of the system. From Eqs. (53) and (54) we can see that both the local magnetization and the spin-pair-correlation function consist of two parts: one decays with  $t$ , and the other vibrates with  $t$ . Since  $\tau$  increases to infinity as the temperature approaches the static critical point  $T_c$  ( $K_c = J/k_\beta T_c = b/2d$ ), the decay term will occur critical slowing down phenomenon.

We now turn on the response of the system to the time-dependent external field. According to the general theory of linear response, the complex susceptibility  $\chi(\omega, \mathbf{q})$  is expressed in terms of the equilibrium correlation of magnetization, namely [22]

$$\chi(\omega, \mathbf{q}) = \chi(0, \mathbf{q}) - \frac{i\omega}{k_\beta T} \int_0^\infty \langle M(-\mathbf{q}, 0) M(\mathbf{q}, t) \rangle_e e^{-i\omega t} dt, \quad (56)$$

where

$$\chi(0, \mathbf{q}) = \frac{1}{k_\beta T} \langle M(-\mathbf{q}, 0) M(\mathbf{q}, 0) \rangle_e, \quad (57)$$

and  $\langle \dots \rangle_e$  denotes the average over equilibrium distribution. Because

$$\chi(\omega, \mathbf{q}) = \frac{N\beta}{b} \cdot \frac{1}{1 - \frac{2K}{b} \sum_{i=1}^d \cos(q_i a) + i\omega}, \quad (58)$$

Eqs. (56) and (57), therefore, mean that

$$\langle M(-\mathbf{q}, 0) M(\mathbf{q}, t) \rangle_e = \langle M(-\mathbf{q}, 0) M(\mathbf{q}, 0) \rangle_e \exp\left(-\frac{t}{\tau_{\mathbf{q}}}\right), \quad (59)$$

and

$$\begin{aligned} \langle M(-\mathbf{q}, 0) M(\mathbf{q}, 0) \rangle_e &= \frac{N}{b \left[ 1 - \frac{1}{b} K(\mathbf{q}) \right]} \\ &= \frac{N}{b \left[ 1 - \frac{2K}{b} \sum_{i=1}^d \cos(q_i a) \right]}, \quad (60) \end{aligned}$$

where

$$\tau_{\mathbf{q}} = \frac{1}{1 - \frac{1}{b} K(\mathbf{q})} = \frac{1}{1 - \frac{2K}{b} \sum_{i=1}^d \cos(q_i a)}. \quad (61)$$

It is interesting to note that as the temperature approaches the static critical point  $T_c$ , for  $\mathbf{q} = 0$  the static spatial correlation diverges, while for  $\mathbf{q} \neq 0$  it remains finite. However, whether

the singularity occurs or not, the dynamic responses strongly depend on the frequency  $\omega$  and the wave vector  $\mathbf{q}$  of the external field.

#### IV. CONCLUDING REMARKS

In this paper, the single-spin transition Glauber dynamics is used to investigate the kinetic Gaussian model in a periodic external field. We have exactly obtained the local magnetization and equal-time pair-correlation function of the  $d$ -dimensional isotropic hypercubic lattice Gaussian model by using Fourier's transformation. The related critical dynamics characteristics of the system are discussed.

The master equation (1) with the transition probability given by Eq. (11) conserves the important features of a cooperative system. When the system is in a time-dependent external field, the dynamical model itself cannot be exact at very high frequencies, it is only suitable for the case of low frequencies. In fact, the existence of a high-frequency field weakens stochastic motion and makes a thermal equilibrium state with canonical distribution impossible. Even so, it would not bring any impact for the characteristic behavior of the system at low frequencies. The present work has clearly shown that the local magnetization, the equal-time pair cor-

relation, and the dynamical responses of the system to a time-dependent external field strongly depend on the frequency and the wave vector of the external field, and when  $\omega \rightarrow 0$  and  $\mathbf{q} \rightarrow \mathbf{0}$ , they approach static results.

The Gaussian model is certainly an idealization, but it is interesting and simple enough to obtain some fundamental knowledge of dynamical process in cooperative systems. Furthermore, although it is an extension of Ising model, the Gaussian model is quite different from Ising model in the properties of the phase transition [24]. It is well known that in the equilibrium case the Gaussian model is exactly solvable on translational invariant lattices [25]. Meantime, as we have done, the Gaussian model is also exactly solvable in dynamical case [20]. Finally, we can anticipate that the kinetic Gaussian model will be a starting point to study the kinetic  $s^4$  (or  $\phi^4$ ) model.

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- [1] R. J. Glauber, *J. Math. Phys.* **4**, 294 (1963).
  - [2] K. Kawasaki, *Phys. Rev.* **145**, 224 (1966).
  - [3] K. Kawasaki, in *Phase Transition and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1972), Vol. 2.
  - [4] S. K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, 1976).
  - [5] P. C. Hohenberg and B. I. Halperin, *Rev. Mod. Phys.* **49**, 435 (1977).
  - [6] R. J. Myerson, *Phys. Rev. B* **14**, 4136 (1976).
  - [7] B. I. Halperin, P. C. Hohenberg, and S. K. Ma, *Phys. Rev. Lett.* **29**, 1548 (1972); *Phys. Rev. B* **10**, 139 (1974).
  - [8] Y. Achiam and J. M. Kosterlitz, *Phys. Rev. Lett.* **41**, 128 (1978); Y. Achiam, *J. Phys. A* **13**, 1825 (1980); *Phys. Rev. B* **31**, 4732 (1985); **32**, 1796 (1985); **33**, 7762 (1986).
  - [9] J. O. Indekeu, A. L. Stella, and J. Rogiers, *Phys. Rev. B* **32**, 7333 (1985).
  - [10] E. J. Lage, *J. Phys. A* **18**, 2289 (1985); **18**, 2411 (1985); *Phys. Lett. A* **127**, 9 (1988).
  - [11] D. Kandel, *Phys. Rev. B* **38**, 486 (1988).
  - [12] J. Zhou and Z. R. Yang, *Phys. Rev. B* **39**, 9423 (1989); Z. R. Yang, **46**, 11 578 (1992).
  - [13] M. D. Lacasse, J. Vinals, and M. Grant, *Phys. Rev. B* **47**, 5646 (1993).
  - [14] B. C. S. Grandi and W. Figueiredo, *Phys. Rev. E* **54**, 4722 (1996).
  - [15] J. Rogiers and J. O. Indekeu, *Phys. Rev. B* **41**, 6998 (1990).
  - [16] J. Wang, *Phys. Rev. B* **47**, 869 (1993).
  - [17] P. Q. Tong, *Phys. Rev. E* **56**, 1371 (1997).
  - [18] F. G. Wang and C. K. Hu, *Phys. Rev. E* **56**, 2310 (1997).
  - [19] A. Vespignani, S. Zapperi, and V. Loreto, *J. Stat. Phys.* **88**, 47 (1997).
  - [20] J. Y. Zhu and Z. R. Yang, *Phys. Rev. E* **59**, 1551 (1999).
  - [21] S. W. Sides, P. A. Rikvold, and M. A. Novotny, *Phys. Rev. E* **57**, 6512 (1998).
  - [22] M. Suzuki and R. Kubo, *J. Phys. Soc. Jpn.* **24**, 51 (1968).
  - [23] T. H. Berlin and M. Kac, *Phys. Rev.* **86**, 821 (1952).
  - [24] S. Li and Z. R. Yang, *Phys. Rev. E* **55**, 6656 (1997).
  - [25] H. E. Stanley, *Introduction to Phase Transition and Critical Phenomena* (Oxford University Press, Oxford, 1983).